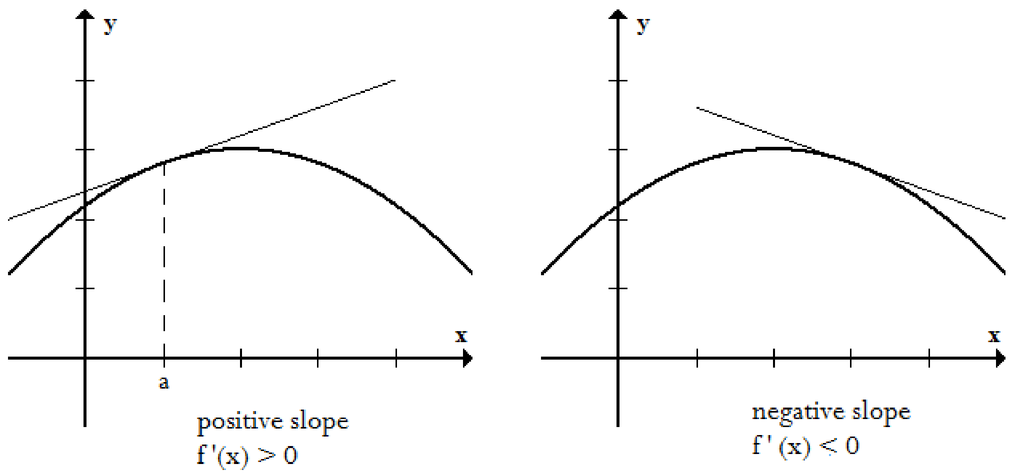
3.3 Using Derivatives to analyze a Function

What does f' say about f? It tells us about the slope of f. But even more, it tells us when $f(x)$ is increasing or decreasing. This is extremely useful when trying to figure out what the graph looks like.



Let's go over this with a bit more formal language.

1. If $f'(x) > 0$ on an interval I, then f is **increasing**.

2. If $f'(x) < 0$ on an interval I, then f is **deceasing**.

I'd like to go over the proof for increasing. The proof for the decreasing part is similar.

Proof. Suppose that $f'(x) > 0$ for all x in the interval I. The Mean Value Theorem tells us that for any two points in I (let's call them x_1 and x_2) where $x_1 < x_2$, there exists a c in I, such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since c is between x_1 and x_2, then $f'(c) > 0$. Plus, $(x_2 - x_1) > 0$. Therefore,

$$f(x_2) - f(x_1) > 0$$

By definition, this means $f(x)$ is increasing.

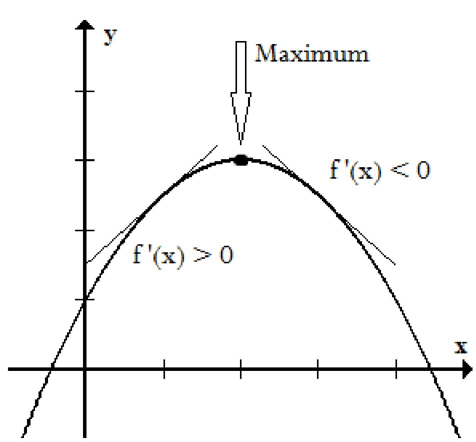
□

3.3.1 First Derivative Test

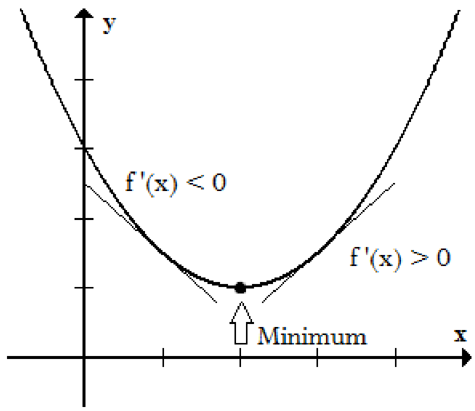
Recall that when $f'(c) = 0$ or $f'(c)$ does not exist, then $x = c$ is called a **critical value**.

Suppose c is a critical value of a continuous function. This means $f'(c) = 0$.

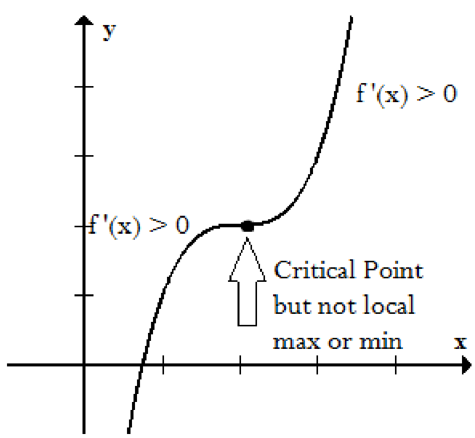
1. If $f'(x)$ changes from a positive $(+)$ to a negative $(-)$ at $x = c$, then $f(x)$ has a local maximum at $x = c$.



2. If $f'(x)$ changes from a positive $(-)$ to a negative $(+)$ at $x = c$, then $f(x)$ has a local minimum at $x = c$.



3. If $f'(x)$ does not change signs, f has neither a max nor min at $x = c$



Example 3.9. Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing / decreasing.

1. Find all critical values.

(a) Find $f'(x)$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

(b) Find where $f'(x)$ does not exist.

Since, $f'(x)$ exists everywhere, we do not get any critical values from this.

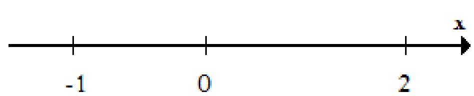
(c) Set $f'(x) = 0$ for other critical values.

$$12x^3 - 12x^2 - 24x = 0$$
$$12x(x^2 - x - 2) = 0$$
$$12x(x-2)(x+1) = 0$$

We have critical values at $x = 0, -1, 2$.

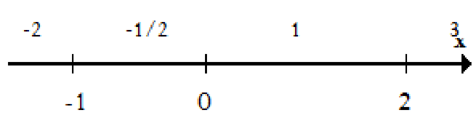
2. Use a number line to determine increasing and decreasing intervals.

(a) Draw a number line with all the critical values plotted.



(b) Pick a point in each region. Plug that number into $f'(x)$ to determine if it's positive $(+)$ or negative $(-)$.

I'll choose $x = -2, -1/2, 1, 3$



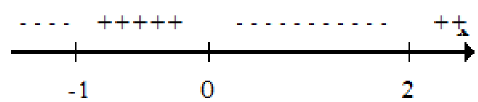
$$f'(-2) < 0$$

$$f'(-1/2) > 0$$

$$f'(1) < 0$$

$$f'(3) > 0$$

(c) Mark this on the number line



This shows $f(x)$ is

(a) decreasing on the intervals $(-\infty, -1)$ and $(0, 2)$

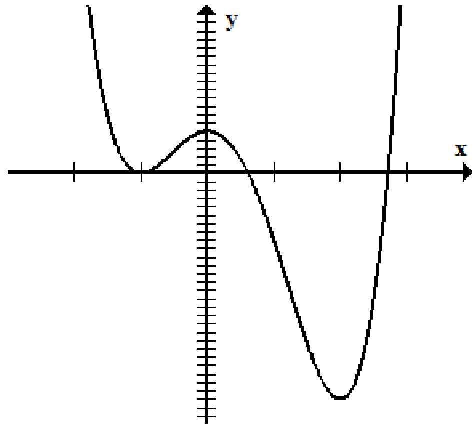
(b) increasing on the intervals $(-1, 0)$ and $(2, \infty)$.

Based on the First Derivative Test, $f(x)$ has a maximum $x = 0$. The point is $(0, 5)$. When we ask for the local maximum, it's better to list it as a point.

There are two local minimums. One is at $x = -1$. The other is at $x = 2$. As points, the local minimums are $(-1, 0)$ and $(2, -27)$.

Remember, you need to plug $x = -1$ and $x = 2$ into the original $f(x)$ to get the y-value.

Looking at the graph, it appears our analysis is correct.



Example 3.10. Find all local max/mins of $f(x) = \cos^2(x) + \sin(x)$ on the interval $[0, \pi]$.

1. Find all critical values.

   (a) Find $f'(x)$

   $$f'(x) = -2\cos(x) \cdot \sin(x) + \cos(x)$$

   (b) Find where $f'(x)$ does not exist.

   Since, $f'(x)$ exists everywhere, we do not get any critical values from this.

   (c) Set $f'(x) = 0$ for other critical values.

   $$-2\cos(x) \cdot \sin(x) + \cos(x) = 0$$

   $$-2\cos(x)\,(\sin(x) - 1/2) = 0$$

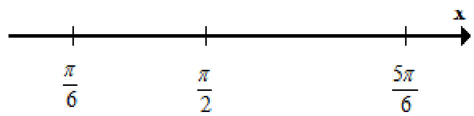
   $$-2\cos(x) = 0 \text{ or } \sin(x) = 1/2$$

   $-2\cos(x) = 0$ gives us $x = \pi/2$.

$\sin(x) = 1/2$ gives us $x = \pi/6$ and $x = 5\pi/6$.

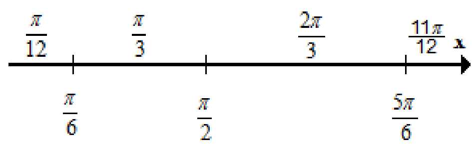
2. Use a number line to determine increasing and decreasing intervals. e

   (a) Draw a number line with all the critical values plotted.



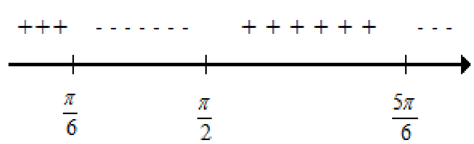
   (b) Pick a point in each region. Plug that number into $f'(x)$ to determine if it's positive (+) or negative (−).

   I'll choose $x = \pi/12, \pi/3, 2\pi/3, \pi$



$$f'(\pi/12) > 0$$
$$f'(\pi/3) < 0$$
$$f'(2\pi/3) > 0$$
$$f'(11\pi/12) < 0$$

   (c) Mark this on the number line



This shows $f(x)$ is

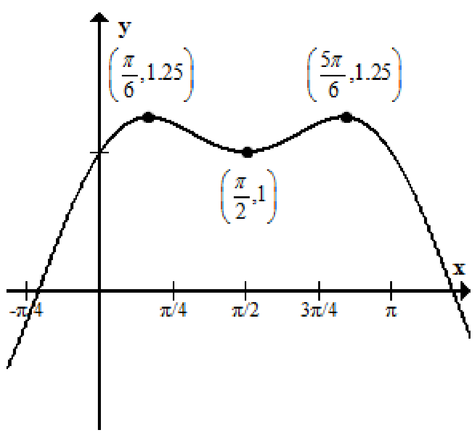
(a) decreasing on the intervals $(\pi/6, \pi/2)$ and $(5\pi/6, \pi)$

(b) increasing on the intervals $(0, \pi/6)$ and $(\pi/2, 5\pi/6)$.

Based on the First Derivative Test, $f(x)$ has a maximum $x = \pi/6$ and $x = 5\pi/6$. The points are $(\pi/6, 1.25)$ and $(5\pi/6, 1.25)$. When we ask for the local maximum, it's better to list it as a point.

The local minimum is at $x = \pi/2$. As a point, the local minimum is $(\pi/2, 1)$.

Looking at the graph of $f(x) = \cos^2(x) + \sin(x)$, we see that our analysis is correct.



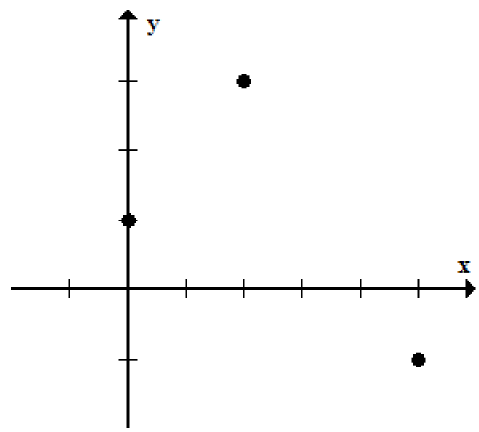
Example 3.11. Sketch a graph satisfying the following conditions.

1. $f(0) = 1$
2. $f(2) = 3$
3. $f(5) = -1$

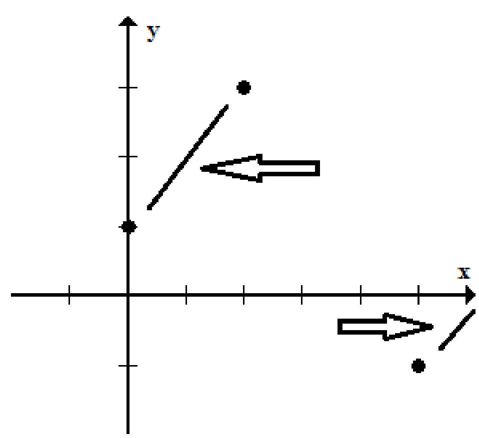
4. $f'(x) > 0$ on $(0, 2)$ and $(5, \infty)$

5. $f'(x) < 0$ on $(-\infty, 0)$ and $(2, 5)$

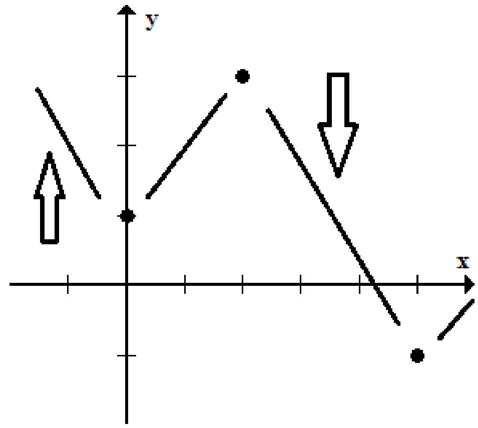
Ok, so the first three conditions are just points. Just like when we sketched graphs with limits, we'll follow the same procedure here. Let's plot those three points.



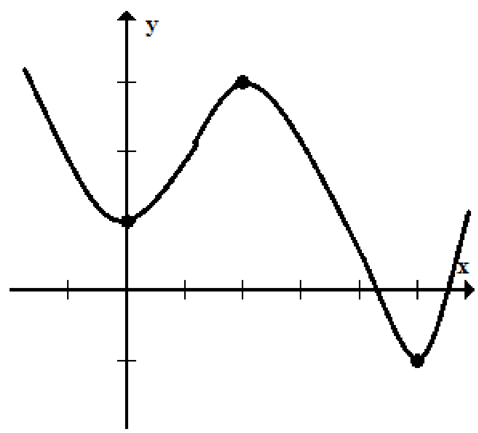
The fourth condition: $f'(x) > 0$ means the function is increasing. So let's place an increasing line in those regions.



The fifth condition: $f'(x) < 0$ means the function is decreasing. So let's place a decreasing line in those regions.



You can connect all these together with a smooth curve.



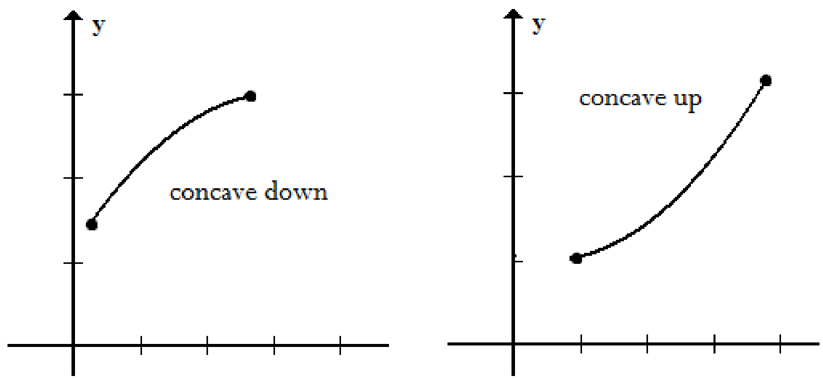
Now how did I know how to connect the pionts. Originally, I connected them with a straight line. Then I connected them with a curved line. There are two ways of connecting two points with a curved line. We get this information from the second derivative.

Recall, a derivative will determine when the original function is increasing or decreasing.

So a second derivative will determine if the first derivative (slopes) are getting increasing or decreasing.

3.3.2 Concavity and Inflection Points

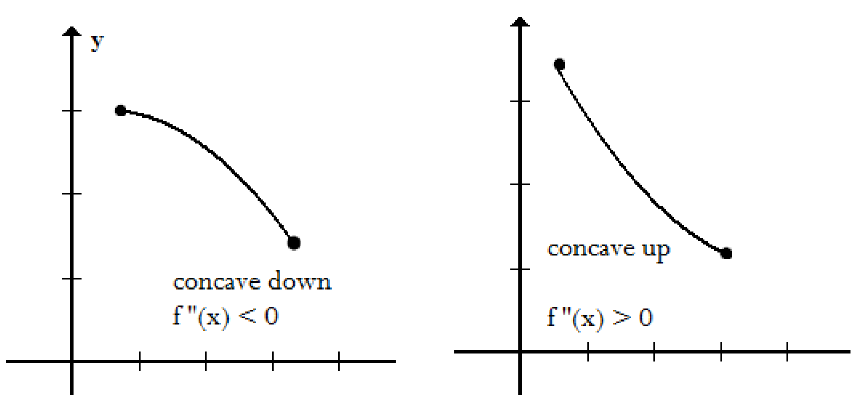
Let's take a look at a region where $f(x)$ is increasing. There are two ways of connecting these points.



Notice that in the first graph, the slopes are getting smaller (decreasing). This means $f''(x) < 0$. We call this concave down

In the second graph, the slopes are increasing. This means $f''(x) > 0$. We call this concave up.

Now let's look at two decreasing lines and how concavity works.

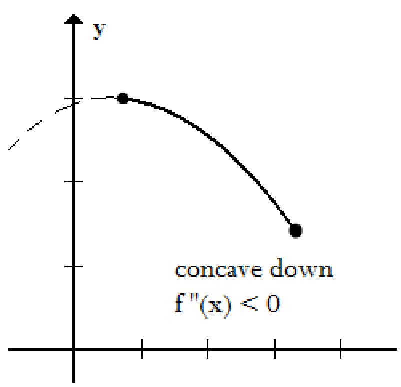


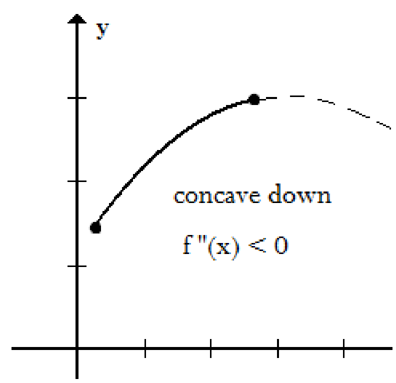
In the first graph, the slopes are decreasing (remember, $-9 < -1$). So larger negative numbers are actually "smaller." Since $f''(x) < 0$, we call this concave down.

In the second graph, the slopes are increasing. Note, a slope of -1 is actually larger than -9, since you had to increase from -9 to -1 on the number line. Since $f''(x) > 0$, we call this concave up.

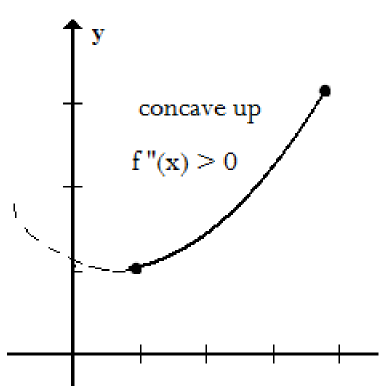
I tend to remember concave up or down based on parabolas. From algebra, we learned that a parabola can open "up" or "down." Concavity works exactly like that.

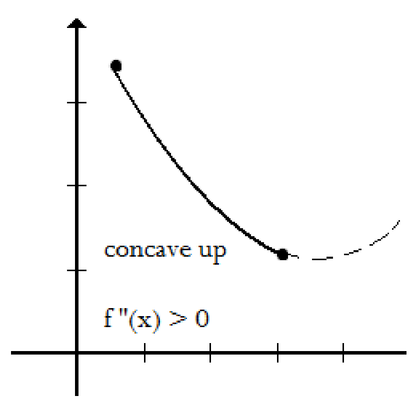
Take a look at the two concave down graphs. If you extend their lines, they look like a parabola opening down (i.e., concave down).



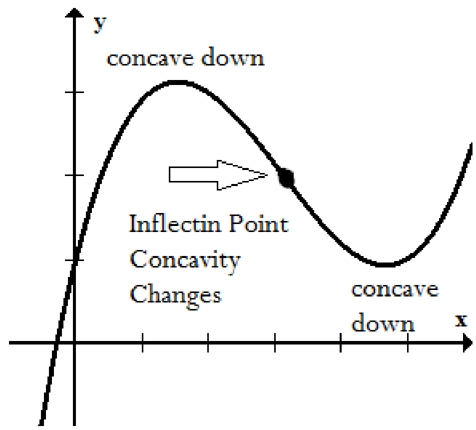


You can see it's very similar for concave up. Concave up graphs look very similar to parabolas opening up.





Definition 3.2 (Inflection Point)**.** A point on $f(x)$ is called an **inflection point** if $f(x)$ is continuous there, and it changes concavity.



Now it's time to sketch some graphs. We'll start with some easy ones and work our way to the more challenging ones.

Example 3.12. Let $f(x) = 2x^3 - 3x^2 - 12x$.

- Find where f is increasing / decreasing.
- Find all local maximums and minimums
- Find intervals of concavity and inflection points
- Use all the above information to sketch the graph

1. First, we need to find the derivative

$$f'(x) = 6x^2 - 6x - 12$$

Next, figure out when $f'(x) = 0$ or when $f'(x)$ does not exist.

Since $f'(x)$ exists everywhere, we'll just go ahead and solve

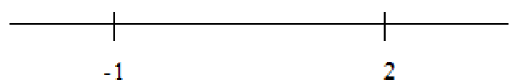
$$6x^2 - 6x - 12 = 0$$

$$6(x^2 - x - 2) = 0$$

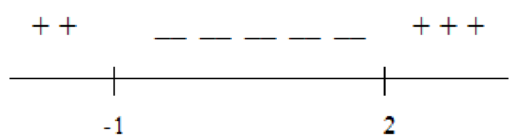
$$6(x-2)(x+1) = 0$$

Solving this, we get critical values at $x = -1$ and $x = 2$.

2. Next, we set up the number line with our critical values.



3. Pick a number from each region, determine if $f'(x)$ is positive or negative. I'll let you plug in any number from each region. If you do it correctly, you get the following



So $f(x)$ is

(a) Increasing on the intervals $(-\infty, -1)$ and $(2, \infty)$

(b) Decreasing on the interval $(-1, 2)$

4. We see from the number line we have a local maximum at $x = -1$ and a local minimum at $x = 2$. Write these as points (since we will have to plot them soon).

$$\text{Local Max: } (-1, 7)$$

$$\text{Local Min: } (2, -20)$$

5. Find the intervals of concavity. To do this, we need to find $f''(x)$.

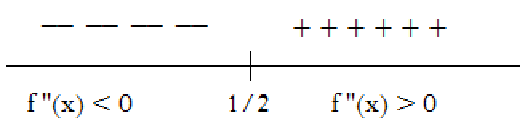
$$f''(x) = 12x - 6$$

We need to find the "critical values" to $f''(x)$.

$$12x - 6 = 0$$

Solving this, we get $x = 1/2$.

6. We use the number line, like we did with the first derivative, to find when $f''(x) > 0$ or when $f''(x) < 0$.



So $f(x)$ is concave down on the interval $(-\infty, 1/2)$

and $f(x)$ is concave up on the interval $(1/2, \infty)$

7. To find the inflection point, we need to satisfy two conditions.

   (a) Does concavity change?

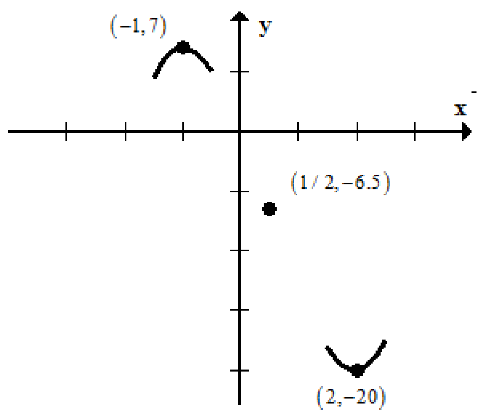
   Yes, concavity changes at $x = 1/2$.

   (b) Does the point exist at $x = 1/2$?

   Yes, plug $x = 1/2$ into $f(x)$ and we get the point $(1/2, -6.5)$.

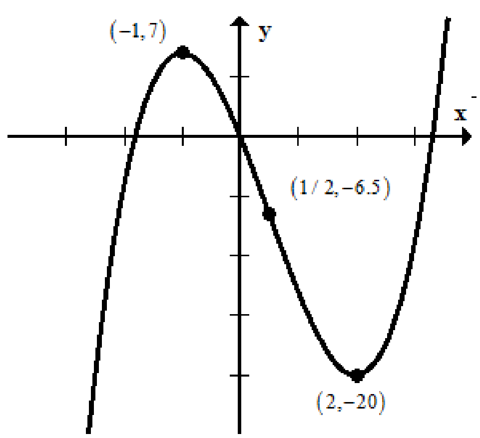
8. Now it's time to put all of this together to sketch the graph.

   (a) Plot all important points (local max/min, inflection points, intercepts, etc.)



   Note, I also added a curve at the local max and min. I do this because it reminds me that they are local max / mins.

   (b) Lastly, I connect all the points using the appropriate concavity.



Example 3.13. Let $f(x) = x^4 - 4x^3$.

- Find where f is increasing / decreasing.
- Find all local maximums and minimums

- Find intervals of concavity and inflection points
- Use all the above information to sketch the graph

1. First, we need to find the derivative

$$f'(x) = 4x^3 - 12x^2$$

Next, figure out when $f'(x) = 0$ or when $f'(x)$ does not exist.

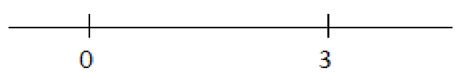
Since $f'(x)$ exists everywhere, we'll just go ahead and solve

$$4x^3 - 12x^2 = 0$$

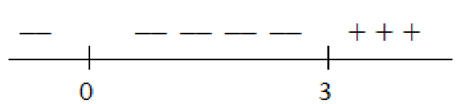
$$4x^2(x-3) = 0$$

Solving this, we get critical values at $x = 0$ and $x = 3$.

2. Next, we set up the number line with our critical values.



3. Pick a number from each region, determine if $f'(x)$ is positive or negative. I'll let you plug in any number from each region. If you do it correctly, you get the following



So $f(x)$ is

(a) Increasing on the intervals $(-\infty, 0)$ and $(0, 3)$. In this case, you can also write $(-\infty, 3)$.

(b) Decreasing on the interval $(3, \infty)$

4. We see from the number line we have a local maximum at $x = -1$ and a local minimum at $x = 2$. Write these as points (since we will have to plot them soon).

Local Max: None

Local Min: $(3, -27)$

5. Find the intervals of concavity. To do this, we need to find $f''(x)$.

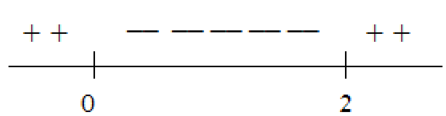
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

We need to find the "critical values" to $f''(x)$.

$$12x(x-2) = 0$$

Solving this, we get $x = 0$ and $x = 2$. These are our possible points of inflection.

6. We use the number line, like we did with the first derivative, to find when $f''(x) > 0$ or when $f''(x) < 0$.



So $f(x)$ is concave down on the interval $(0, 2)$

and $f(x)$ is concave up on the interval $(-\infty, 0)$ and $(2, \infty)$

7. To find the inflection point, we need to satisfy two conditions.

   (a) Does concavity change?

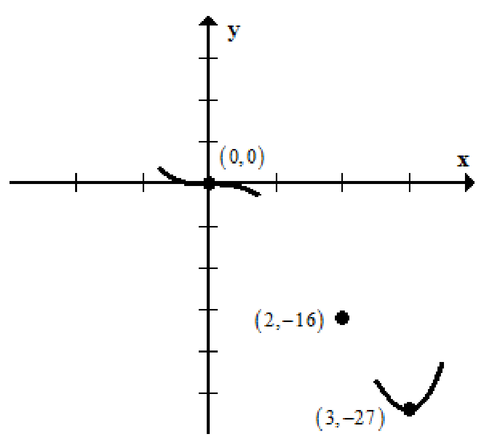
   Yes, concavity changes at $x = 0$ and $x = 2$.

   (b) Does the point exist at $x = 0$ or $x = 2$?

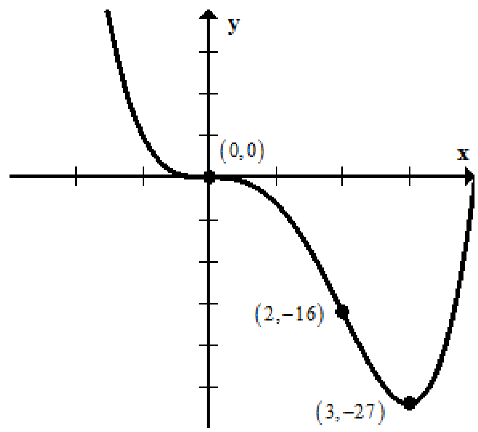
   Yes, we get the following points: $(0, 0)$ and $(2, -16)$

8. Now it's time to put all of this together to sketch the graph.

   (a) Plot all important points (local max/min, inflection points, intercepts, etc.)



   (b) Lastly, I connect all the points using the appropriate concavity.



Example 3.14. Sketch $f(x) = x^{1/3}(x+4)$.

- Find where f is increasing / decreasing.
- Find all local maximums and minimums
- Find intervals of concavity and inflection points
- Use all the above information to sketch the graph

1. First, we need to find the derivative

$$f'(x) = x^{1/3} \cdot (1) + (x+4) \cdot \frac{1}{3}x^{-2/3}$$

Next, figure out when $f'(x) = 0$ or when $f'(x)$ does not exist. Before doing that, we need to simplify this. This means ALGEBRA. If you cannot do this now, make sure you can by exam time!

$$\begin{aligned}
f'(x) &= x^{1/3}\cdot(1) + (x+4)\cdot\frac{1}{3}x^{-2/3} \\
&= x^{1/3} + \frac{1}{3}x^{-2/3}(x+4) \\
&= x^{-2/3}\left(x + \frac{1}{3}(x+4)\right) \\
&= x^{-2/3}\left(\frac{4}{3}x + \frac{4}{3}\right) \\
&= \frac{4}{3}x^{-2/3}(x+1) \\
&= \frac{4(x+1)}{3x^{2/3}}
\end{aligned}$$

$f'(x)$ does not exist at $x = 0$, since it makes the denominator equal to 0.

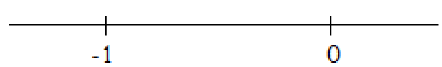
Next we set $f'(x) = 0$. We do that by setting the numerator equal to 0.

$$\frac{4(x+1)}{3x^{2/3}} = 0$$

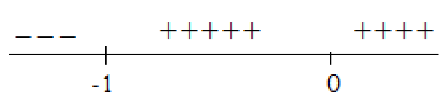
$$4(x+1) = 0$$

Solving this, we get critical values at $x = -1$.

2. Next, we set up the number line with our critical values.



3. Pick a number from each region, determine if $f'(x)$ is positive or negative. I'll let you plug in any number from each region. If you do it correctly, you get the following



So $f(x)$ is

(a) Increasing on the intervals $(-1, 0)$ and $(0, \infty)$. In this case, you can also write $(-1, \infty)$. You need to make sure $x = 0$ is in the domain (which it is).

(b) Decreasing on the interval $(-\infty, -1)$

4. We see from the number line we have a local minimum at $x = -1$ and no local maximum.

Local Max: None

Local Min: $(-1, -3)$

5. Find the intervals of concavity. To do this, we need to find $f''(x)$. Let's use $f'(x) = \frac{4}{3}x^{-2/3}(x+1)$.

$$f''(x) = \frac{4}{3}x^{-2/3} \cdot (1) + (x+1) \cdot -\frac{8}{9}x^{-5/3}$$

$$f''(x) = \frac{4}{9}x^{-5/3}(3x - 2(x+1))$$

$$f''(x) = \frac{4}{9}x^{-5/3}(x-2)$$

$$f''(x) = \frac{4(x-2)}{9x^{5/3}}$$

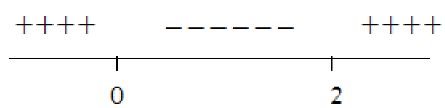
We need to find the "critical values" to $f''(x)$.

$$\frac{4(x-2)}{9x^{5/3}} = 0$$

Solving this, we get $x = 2$.

Our other critical value is $x = 0$ because that's when $f''(x)$ does not exist.

6. We use the number line, like we did with the first derivative, to find when $f''(x) > 0$ or when $f''(x) < 0$.



So $f(x)$ is concave down on the interval $(0, 2)$

and $f(x)$ is concave up on the interval $(-\infty, 0)$ and $(2, \infty)$

7. To find the inflection point, we need to satisfy two conditions.

   (a) Does concavity change?

   Yes, concavity changes at $x = 0$ and $x = 2$.

   (b) Does the point exist at $x = 0$ or $x = 2$?

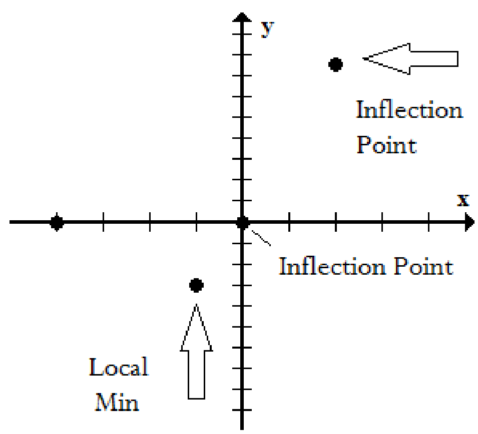
   Yes, we get the following points: $(0, 0)$ and $(2, 7.56)$

8. We should probably find some other points too. We should try to find x and y intercepts. Typically we do this at the beginning of the problem.

- x-intercepts: $(0, 0)$ and $(-4, 0)$
- y-intercept: $(0, 0)$

Now it's time to put all of this together to sketch the graph.

(a) Plot all important points (local max/min, inflection points, intercepts, etc.)



(b) Lastly, I connect all the points using the appropriate concavity.