4.4 Indefinite Integrals

Section 4.3 gave us our main tool for evaluating integrals, as long as we can find a function whose derivative gives us the integrand. Here, we run through a collection of those functions and state the previous Part 2 of the FTC so that we can apply it to more application-type problems.

We have the connection between integration and differentiation – they are inverse operations – but we still have clunky notation. In order to fix this, we define the

Definition 4.4 (Indefinite integral)**.**

$$\int f(x)dx = F(x)$$

such that $F'(x) = f(x)$.

For example, since

$$\frac{d}{dx}\left[x^9\right] = 9x^8,$$

then

$$\int 9x^8\ dx = x^9 + C,$$

where C is some constant, as we discussed in earlier sections. Also, since

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x),$$

then

$$\int \cos(x)\ dx = \sin(x) + C.$$

Because of these $+C$'s, we don't get out just one single function, but a whole family of functions – one function for each possible value of C. Now, there is something to be careful about. When we talked about definite integrals, $\int_a^b f(x)dx$, the result was always a number.

And it is, for definite integrals. For indefinite integrals, since there are no bounds of integration, the result is always a function of the integrating variable.

We now give a table of integration formulas, which really is the same as the table of anti-differentiation formulas we saw in Section 4.9. We can verify any of the formulas by differentiating the right side to obtain the function inside the integral on the left side.

Integral
$\int cf(x)\, dx = c\int f(x)dx$
$\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
$\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c,\, n \neq -1$
$\int \sin(x)\, dx = -\cos(x) + c$
$\int \cos(x)\, dx = \sin(x) + c$
$\int k\, dx = kx + c$
$\int \sec^2(x)\, dx = \tan(x) + c$
$\int \sec(x)\tan(x)\, dx = \sec(x) + c$
$\int \csc^2(x)\, dx = -\cot(x) + c$
$\int \csc(x)\cot(x)\, dx = -\csc(c) + c$

From these formulas, it should be clear that the most general form of an integral has an arbitrary constant added to the end of it. This will need to be done for ALL indefinite integrals you encounter, with no exceptions, period.

Example 4.28. Find the general, indefinite integral

$$\int 15x^4 + 8x^3 - \cos(x)dx.$$

We can break this into a sum/difference of three integrals, and evaluate them each according to the rules in the previous table:

$$\begin{aligned}\int 15x^4 + 8x^3 - \cos(x)\ dx =& 15\int x^4\ dx + 8\int x^3\ dx - \int \cos(x)\ dx\\ =&\frac{15}{5}x^5 + \frac{8}{4}x^4 - (\sin(x)) + c\\ =&3x^5 + 2x^4 - \sin(x) + c\end{aligned}$$

If we wish to check our answer, we can differentiate it. Remember, that since c is a constant, $\frac{d}{dx}[c] = 0$:

$$\frac{d}{dx}\left[3x^5 + 2x^4 - \sin(x) + c\right] = 3\cdot 5x^4 + 2\cdot 4x^3 - \cos(x) + 0 = 15x^4 + 8x^3 - \cos(x).$$

Example 4.29. Evaluate

$$\int \frac{\sin(x)}{\cos^2(x)}dx.$$

So far, there is no rule for integrating a quotient of two functions, so we cannot integrate as is. Instead, use some trig identities to simplify and try to get one of the known forms in the table:

$$\begin{aligned}\int \frac{\sin(x)}{\cos^2(x)}dx &= \int \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)}dx\\ &= \int \tan(x)\sec(x)dx\\ &= \sec(x) + c\end{aligned}$$

Example 4.30. We can also use our handy table for definite integrals: evaluate

$$\int_{\pi}^{2\pi} 3x^4 + \cos(x)\ dx$$

We have, from the formulas in our table,

$$\begin{aligned}\int_{\pi}^{2\pi} 3x^4 + \cos(x)\ dx =& \frac{3}{5}x^5 + \sin(x))\ |_{\pi}^{2\pi} \\ =& \left(\frac{3}{5}(2\pi)^5 + \sin(2\pi)\right) - \left(\frac{3}{5}(\pi)^5 + \sin(\pi)\right) \\ =& \frac{3\pi^5}{5}(32-1) \\ =& \frac{91\pi^5}{5}\end{aligned}$$

Example 4.31. Evaluate

$$\int \frac{2y^2 - 3\sqrt{y} + 7y^{10}}{y^2} dy.$$

First, the variable of integration is y – treat is no differently than any other, it's just a variable. Second, we have a quotient again, and we cannot integrate that. All we can do is some algebra first, to get rid of that quotient:

$$\begin{aligned}\int \frac{y - 3\sqrt{y} + 7y^{10}}{y^2} dy =& \int \frac{2y^2}{y^2} dy - 3\int \frac{y^{1/2}}{y^2} dy + 7\int \frac{y^{10}}{y^2} dy \\ =& \int 2dy - 3\int y^{-3/2} dy + 7\int y^8 dy \\ =& 2y - \frac{3}{-1/2} y^{-1/2} + \frac{7}{9} y^9 + c \\ =& 2y + 6y^{-1/2} + \frac{7}{9} y^9 + c\end{aligned}$$

4.4.1 Applications

We know that given a function $f(x)$, $f'(x)$ represents the rate of change of $y = f(x)$ with respect to x and that $f(b) - f(a)$ is the total change in y when x changes from a to b – note that this is the net change, not the distance traveled. We could increase, decrease and increase again and though we move a lot, the net change could be quite low. The FTC is very much this net change result:

Definition 4.5 (The Net Change Theorem)**.** The integral of a rate of change is the net change in the function:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

There are all kinds of examples of this; we give some common ones now, for future problem use:

- If $V(t)$ is the volume of water at time t, then $V'(t)$ is the change in volume. The net change of our container of water is

$$\int_a^b V'(t)dt = V(b) - V(a),$$

  from starting time $t = a$ to ending time $t = b$.

- If the mass of a rod from one end to a point in the middle is $m(x)$, then we have the linear density as $\rho(x) = m'(x)$. Thus,

$$\int_a^b \rho'(x)dx = m(b) - m(a)$$

  is the mass of the rod between $x = a$ and $x = b$.

- If population grows at a rate of $\frac{dP}{dt}$, then

$$\int_a^b \frac{dP}{dt}dt = P(b) - P(a)$$

  is the net change in population from time $t = a$ to $t = b$.

- If an object moves along a straight path by the function $s(t)$, we know its velocity is $v(t) = s'(t)$, so

$$\int_a^b v(t)\ dt = s(b) - s(a)$$

is the net change in distance the object traveled. If, on the other hand, we want to find the total amount of distance the object has traveled, we have to consider two different problems – the interval when $v(t) \geq 0$ and the interval when $v(t) < 0$. To do this, compute the integral

$$\int_a^b |v(t)| dt.$$

If we think about this in terms of the area under a curve,

$$\int_a^b v(t) dt$$

is the net area, when the area above the x-axis is subtracted by the area below the x-axis. However,

$$\int_a^b |v(t)| dt$$

is the total area between the function and the x-axis, which always must be positive.

- The acceleration of an object at time t is given by $a(t) = v'(t)$, and thus the net change in velocity from $t = a$ to $t = b$ is

$$\int_a^b a(t)\ dt = v(b) - v(a).$$

Example 4.32. An object moves along a line with acceleration given by $a(t) = 2t+1$. Find the displacement of the object from $t = 1$ to $t = 4$ and then find the total distance traveled over the same time, if the velocity at $t = 0$ is -12.

Since we begin with acceleration, we need to integrate to get velocity as a function, then take a definite integral to get the displacement.

$$\begin{aligned} v(t) &= \int a(t)\ dt \\ &= \int 2t + 1dt \\ &= t^2 + t + c \end{aligned}$$

$$\begin{aligned} v(0) = -12 &= (0)^2 + 0 + c \\ c &= -12 \end{aligned}$$

$$\begin{aligned} v(t) &= t^2 + t - 12 \\ s(4) - s(1) &= \int_1^4 t^2 + t - 12dt \\ &= \frac{1}{3}t^3 + \frac{1}{2}t^2 - 12t\Big|_1^4 \\ &= \left(\frac{64}{3} + 8 - 48\right) - \left(\frac{1}{3} + \frac{1}{2} - 12\right) \\ &= \frac{-15}{2} \end{aligned}$$

is the total displacement of the object.

In order to find the distance traveled, we need the intervals where the function is positive and where it is negative, To find these, factor $v(t) = t^2 + t - 12 = (t + 4)(t - 3)$, but since time can only be positive, we consider the intervals $(1, 3)$ and $(3, 4)$, as $t = 1$ is the lower bound on the integral and $t = 4$ is the upper bound on the integral. Since $v(t) \leq 0$ for t in $(1, 3)$ by the test-point method, the total distance traveled is

$$\begin{aligned}
\int_1^4 |v(t)|dt &= \int_1^3 -v(t)dt + \int_3^4 v(t)dt \\
&= \int_1^3 -t^2 - t + 12dt + \int_3^4 t^2 + t - 12dt \\
&= \left[\frac{-1}{3}t^3 - \frac{1}{2}t^2 + 12t\right]_1^3 + \left[\frac{1}{3}t^3 + \frac{1}{2}t^2 - 12t\right]_3^4 \\
&= \left(-9 - \frac{9}{2} + 36 + \frac{1}{3} + \frac{1}{2} - 12\right) + \left(\frac{64}{3} + 8 - 48 - 9 - \frac{9}{2} + 36\right) \\
&= -9 - \frac{9}{2} + 36 + \frac{1}{3} + \frac{1}{2} - 12 + \frac{64}{3} + 8 - 48 - 9 - \frac{9}{2} + 36 \\
&= \frac{91}{6}
\end{aligned}$$

Example 4.33. Suppose that the linear density of a rod of length $3\pi/2$ meters is given by

$$\rho(x) = |\sin(x)|.$$

Find the total mass of the rod.

We need to integrate the linear density from one end to the other, from $x = 0$ to $x = 3\pi/2$:

$$\int_0^{3\pi/2} |\sin(x)|\, dx$$

But we need to know when $\sin(x)$ is positive and negative on the interval $[0, 3\pi/2]$. Recall from trig that $\sin(x) > 0$ on the interval $(0, \pi)$ and $\sin(x) < 0$ on the interval $(\pi, 3\pi/2)$.

$$\begin{aligned}
\int_0^{3\pi/2} |\sin(x)| dx &= \int_0^{\pi} \sin(x) dx + \int_{\pi}^{3\pi/2} -\sin(x) dx \\
&= [-\cos(x)]_0^{\pi} + [\cos(x)]_{\pi}^{3\pi/2} \\
&= (-\cos(\pi) + \cos(0)) + (\cos(3\pi/2) - \cos(\pi)) \\
&=(-(-1) + 1) + (0 - (-1)) \\
&=3
\end{aligned}$$