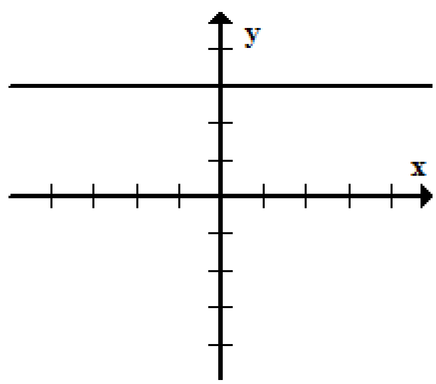
2.3 Differentiation Formulas

In this section we introduce shortcuts to finding derivatives. Up to this point, we had to find a derivative using the limit definition. We will derive the shortcuts using the limit definition. Once we've done that, we will use the shortcuts from then on.

Recall:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Take a look at the graph of $f(x) = 3$ or $y = 3$. This is called constant function.



Without using the derivative, you should be able to see the slope of a constant function, i.e., a horizontal line, is 0. Let's go ahead and prove that.

Theorem 2.1. *If $f(x) = c$, then $f'(x) = 0$.*

We will use the limit definition to derive the conclusion.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

Let's move on to power functions. Recall that a power function is $f(x) = x^n$. Let's take a look at the following table. You can verify these derivatives on your own.

$$\begin{aligned} f(x) &= x^2 & \rightarrow \quad f'(x) &= 2x \\ f(x) &= x^3 & \rightarrow \quad f'(x) &= 3x^2 \\ f(x) &= x^4 & \rightarrow \quad f'(x) &= 4x^3 \\ f(x) &= x^100 & \rightarrow \quad f'(x) &= 100x^99 \end{aligned}$$

Do you see a pattern for the derivative of a power function?

2.3.1 Power Rule

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

To find the derivative of a power function, bring the exponent down in front and subtract the exponent by 1. Let's prove this.

Proof: Let $f(x) = x^n$. We are going to use the other version of the limit definition.

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + xa^{n-2} + a^{n-1})}{x-a} \\
&= \lim_{x\to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + xa^{n-2} + a^{n-1} \\
&= \underbrace{a^{n-1} + a^{n-2}a + a^{n-3}a^2 + ... + aa^{n-2} + a^{n-1}}_{n \text{ terms}} \\
&= a^{n-1} + a^{n-1} + a^{n-1} + ... + a^{n-1} + a^{n-1} \\
&= na^{n-1}
\end{aligned}
$$

Remember that a is just an arbitrary letter to represent an x-value. So if $f'(a) = na^{n-1}$, then we really just showed that $f'(x) = nx^{n-1}$.

The following rule allows us to differentiate any combination of power functions.

2.3.2 Sum / Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f \pm g] = \frac{d}{dx}f \pm \frac{d}{dx}g = f'(x) \pm g'(x)$$

In other words, you can differentiate each term one at a time.

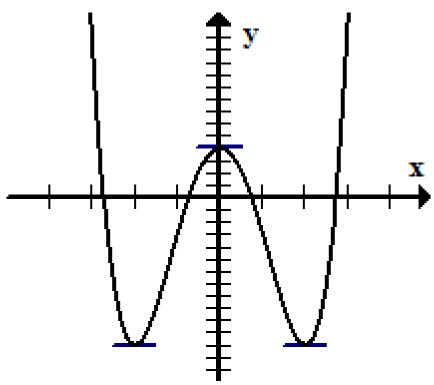
Example 2.11. Find $\frac{d}{dx}[x^8 + 12x^5 - 4x^4 + 10x^3 + 5]$

$$= 8x^7 + 5 \cdot 12x^4 - 4 \cdot 4x^4 + 3 \cdot 10x^3 + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 + 0$$

Example 2.12. Find all points on the curve $y = x^4 - 8x^2 + 4$, where the tangent line is horizontal.

Before doing any calculus work, let's take a look at the graph of $y = x^4 - 8x^2 + 4$.



When it asks you to find all places where the tangent line is horizontal, it's really asking you where is $f'(x) = 0$. From the graph, it appears this happens at $x = -2, 0, 2$. Let's verify that now.

1. Find $f'(x)$

$$\begin{aligned} f'(x) &= 4x^3 - 2 \cdot 8x + 0 \\ &= 4x^3 - 16x \end{aligned}$$

2. To find where you have a slope of 0, set $f'(x) = 0$

$$\begin{aligned} 4x^3 - 16x &= 0 \\ 4x(x^2 - 4) &= 0 \\ 4x(x-2)(x+2) &= 0 \end{aligned}$$

We have a slope of 0 at $x = -2, 0, 2$.

The power rule applies to all numbers for n except $n = 0$.

$$\frac{d}{dx}[x^n] = nx^{n-1} \text{ for all } n \neq 0$$

Example 2.13. Differentiate $f(x) = \sqrt{x}$

1. Rewrite $f(x)$ as x^n

$$f(x) = \sqrt{x} = x^{1/2}$$

2. Now user the power rule to find $f'(x)$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

3. You may rewrite this in radical form or without negative exponents. Sometimes you may have to.

$$f'(x) = \frac{1}{2x^{1/2}} \text{ or } f'(x) = \frac{1}{2\sqrt{x}}$$

Example 2.14. Differentiate $f(x) = \dfrac{5}{8x^7}$

1. Rewrite $f(x)$ as x^n

$$f(x) = \frac{5}{8}x^{-7}$$

2. User the power rule to find $f'(x)$

$$f'(x) = -7 \cdot \frac{5}{8}x^{-8}$$

Remember to subtract 1 from the exponent $\rightarrow -7 - 1 = -8$

3. Clean up the derivative

$$f'(x) = -\frac{35}{8x^8} \text{ or } f'(x) = -\frac{35}{8}x^{-8}$$

Example 2.15. Differentiate $f(x) = -\dfrac{3}{10\sqrt[5]{x}}$

1. Rewrite $f(x)$ as x^n

$$f(x) = -\frac{3}{10}x^{-1/5}$$

2. User the power rule to find $f'(x)$

$$f'(x) = \frac{1}{5} \cdot \frac{10}{3}x^{-6/5}$$

   Note: $-\dfrac{1}{5} - 1 = -\dfrac{6}{5}$

3. Clean up the derivative

$$f'(x) = \frac{2}{3}x^{-6/5}$$

Sometimes you come across derivatives that appear complicated at first, but after simplifying turn out to be fairly nice. Consider the following.

Example 2.16. Let $f(x) = \dfrac{x^3 - 2\sqrt[3]{x}}{x^2}$. Find $f'(x)$.

This is a good practice problem for simplifying. If the denominator has one term, distribute that as a denominator to all the terms in the numerator.

$$f(x) = \frac{x^3 - 2\sqrt[3]{x}}{x^2} = \frac{x^3}{x^2} - \frac{2\sqrt[3]{x}}{x^2}$$

Now simplify each fraction. You probably should rewrite all the terms so they are in the x^n form.

$$f(x) = \frac{x^3}{x^2} - \frac{2x^{1/3}}{x^2}$$
$$f(x) = x - 2x^{-5/3}$$

Now differentiate using the power rule.

$$f'(x) = 1 + \frac{5}{3} \cdot 2x^{-8/3}$$

And clean up

$$f'(x) = 1 + \frac{10}{3}x^{-8/3}$$

So much easier...right?

2.3.3 Product Rule

Now we want to differentiate functions that are defined as a product. For example, $f(x) = (x^2 + 5x + 2)(x^9 - 3x^8)$. Do you see how $f(x)$ is a product of two functions $(x^2 + 5x + 2)$ and $(x^9 - 3x^8)$?

Differentiate using the Product Rule

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Example 2.17. Find $\frac{d}{dx}\left[(x^2 + 5x + 2)(x^9 - 3x^8)\right]$

$$\begin{aligned} y' &= (x^2 + 5x + 2) \cdot \frac{d}{dx}\left[x^9 - 3x^8\right] + \frac{d}{dx}\left[x^2 + 5x + 2\right] \cdot (x^9 - 3x^8) \\ &= (x^2 + 5x + 2) \cdot \left(9x^8 - 24x^7\right) + (2x + 5) \cdot (x^9 - 3x^8) \end{aligned}$$

You can foil and attempt to simplify. For now, I'll leave the answer like this.

One of the biggest mistakes students have with the product rule is the following.

Product Rule - the wrong way

$$\frac{d}{dx}\left[(x^2 + 5x + 2)(x^9 - 3x^8)\right]$$

An incorrect way of using the product rule is to differentiate each factor and then just multiply them together. For example,

$$y' \neq \frac{d}{dx}(x^2 + 5x + 2) \cdot \frac{d}{dx}(x^9 - 3x^8)$$

$$= (2x + 5)(9x^8 - 24x^7)$$

Please do not do this. It's a good way of getting 0 points.

Example 2.18. Find $f'(x)$ when $f(x) = \left(\frac{1}{x^2} + 4x^3 - x^{5/3}\right)\left(3x^{-1/2} - \sqrt[5]{x^2}\right)$

1. Before attempting the product rule, rewrite all terms into the x^n form.

$$f(x) = (x^{-2} + 4x^3 - x^{5/3})(3x^{-1/2} - x^{2/5})$$

If $u = x^{-2} + 4x^3 - x^{5/3}$ and $v = 3x^{-1/2} - x^{2/5}$, then

$$f'(x) = uv' + u'v$$

2. Now use the product rule

$$\begin{aligned} f'(x) &= (x^{-2} + 4x^3 - x^{5/3}) \cdot \frac{d}{dx}\left[3x^{-1/2} - x^{2/5}\right] + \frac{d}{dx}\left[x^{-2} + 4x^3 - x^{-5/3}\right] \cdot (3x^{-1/2} - x^{2/5}) \\ &= (x^{-2} + 4x^3 - x^{5/3})\left(-\frac{3}{2}x^{-3/2} - \frac{2}{5}x^{-3/5}\right) + \left(-2x^{-3} + 12x^2 + \frac{5}{3}x^{-8/3}\right)(3x^{-1/2} - x^{2/5}) \end{aligned}$$

3. You can clean this up a bit but the goal of this section is to show you how to properly use the product rule. Simplifying some of these will come later.

2.3.4 Quotient Rule

If you haven't guessed already, the quotient rule allows us to differentiate functions that look like quotients. For example, we can differentiate $f(x) = \dfrac{3x^3+x}{x^2+10}$

Differentiating using the Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

One way students remember this formula is by remembering this,

$$\frac{d}{dx}\left[\frac{hi}{lo}\right] = \frac{lo\text{-}D\text{-}hi - hi\text{-}D\text{-}lo}{lo^2}$$

If you say it enough times, it should stick. After 10 years, I still say this when using the quotient rule (out loud mostly).

Example 2.19. Find $\dfrac{d}{dx}\left[\dfrac{3x^3+x}{x^2+10}\right]$

1. Make sure all terms are in the x^n form. It appears they are, so let's move on.

2. Use the quotient rule.

   Let $v = 3x^3 + x$ and $u = x^2 + 10$. Then

$$f'(x) = \frac{uv' - vu'}{u^2}$$

$$
\begin{aligned}
f'(x) &= \frac{(x^2+10)\cdot \dfrac{d}{dx}\left[3x^3+x\right] - (3x^3+x)\cdot \dfrac{d}{dx}\left[x^2+10\right]}{\left[x^2+10\right]^2} \\
&= \frac{(x^2+10)\cdot(9x^2+1) - (3x^3+x)(2x)}{(x^2+10)^2}
\end{aligned}
$$

3. You may want practice distributing and simplifying the numerator. At some point we will have to set the numerator equal to 0.

Example 2.20. Differentiate $y = \dfrac{5x^2 - \sqrt[4]{x}}{x^2 - \frac{15}{x^2}}$

1. Change all terms into the x^n form.

$$
y = \frac{5x^2 - x^{1/4}}{x^2 - 15x^{-2}}
$$

2. Use the Quotient Rule

$$
\begin{aligned}
y' &= \frac{(x^2-15x^{-2})\cdot \dfrac{d}{dx}\left[5x^2 - x^{1/4}\right] - (5x^2 - x^{1/4})\cdot \dfrac{d}{dx}\left[x^2 - 15x^{-2}\right]}{(x^2-15x^{-2})^2} \\
&= \frac{(x^2-15x^{-2})\cdot(10x - \frac{1}{4}x^{-3/4}) - (5x^2 - x^{1/4})\cdot(2x+30x^{-3})}{(x^2-15x^{-2})^2}
\end{aligned}
$$

Example 2.21. Find the equation of the tangent line to $y = (1+2x)^2$ at (1,9).

1. To find the slope of the tangent line, we need to find y'.

2. We have two options to find y'.

   (a) Foil and use the power rule $\rightarrow y = (1+2x)^2 = (1+2x)(1+2x) = 1+4x+4x^2$

   $$y' = 4 + 8x$$

   (b) Differentiate using the product rule $\rightarrow y = (1+2x)(1+2x)$

   $$y' = (1+2x)\cdot\frac{d}{dx}(1+2x) + (1+2x)\cdot\frac{d}{dx}(1+2x)$$
   $$y' = (1+2x)(2) + (1+2x)(2)$$
   $$y' = 8x + 4$$

   Either way, we get $y' = 8x + 4$

3. To find the slope at (1,9), we find $f'(1)$

   $$f'(1) = 8(1) + 4 = 12$$

4. Use the point-slope formula to find the equation of the tangent line

   $$y - y_1 = m(x - x_1)$$
   $$y - 9 = 12(x - 1)$$
   $$y = 12x - 3$$

Note, when differentiating, don't use more than one rule. It will take some practice but eventually you'll know which rule to use.

2.3.5 Higher Derivatives

I won't spend too much time on this section. We will do lots of higher derivatives in the coming sections.

Recall

1. $f'(x) =$ first derivative
2. $(f'(x))' = f''(x) =$ second derivative
3. $(f''(x))' = f'''(x) =$ third derivative
4. $f^4(x) =$ fourth derivative
5. $f^{10}(x) =$ tenth derivative

Example 2.22. Let $f(x) = x^3 - 4x^2 + 3x^{1/2} - \dfrac{1}{x} + 4$. Find f' and f''.

1. As always, rewrite so each term is of the form x^n.

$$f(x) = x^3 - 4x^2 + 3x^{1/2} - x^{-1} + 4$$

2. Find f'

$$f'(x) = 3x^2 - 8x + \frac{1}{2}x^{-1/2} + x^{-2} + 0$$

3. Find f''

$$f''(x) = 6x - 8 - \frac{1}{4}x^{-3/2} - 2x^{-3}$$