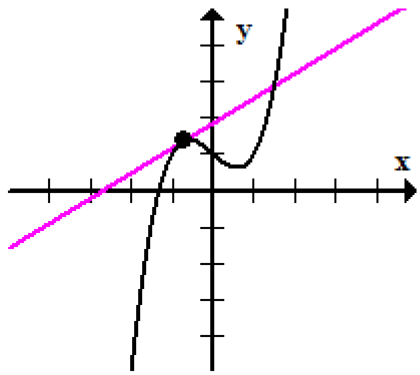
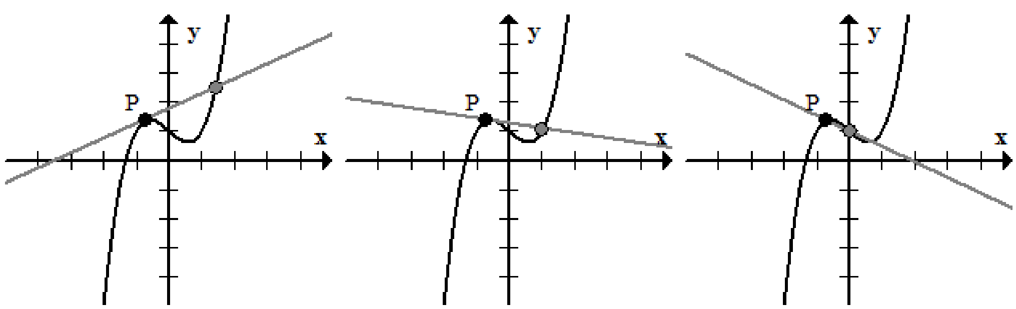
# 2 Derivatives

## 2.1 Derivatives and Rates of Change

Suppose you have a graph of a function. And suppose you want to find a line that touches the graph at a certain point so that the slope of the line is the same as the slope of the graph at that point. It would look something like this:




To find the slope of a tangent (and the slope of a graph at a given point $P$), we find the slopes of many secant lines. What's a secant line you ask? I'll show you.




A secant line is simply a line connecting two points on a graph. So how does that help us find the tangent line? Here's the idea:

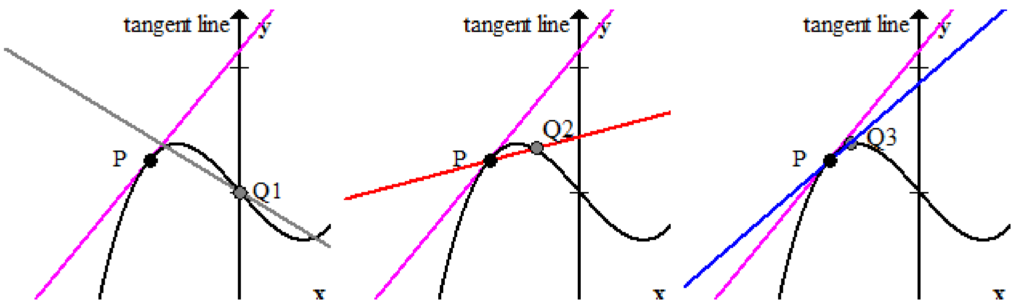
1. Make a secant line with the point $P$ and some other point on the graph (call it $Q1$) and find the slope.

2. Pick a point closer to $P$. Call this $Q2$. Find the slope of this line.

3. Pick a point even closer to $P$. Call it $Q3$. Find the slope.

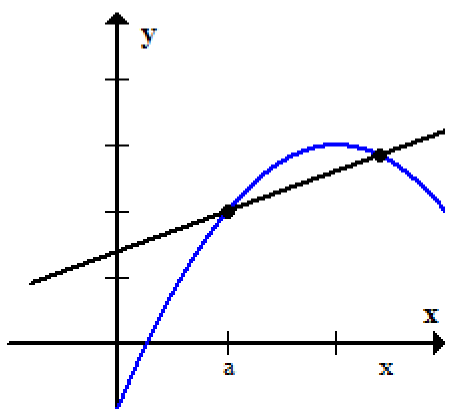
4. Continue this until your second point is extremely close to $P$.

5. The slope of these secant lines should be very close to the slope of the graph at $P$.




Can you see when the point $Q$ gets closer to $P$, the secant lines are getting closer to looking like the tangent line? Since you know how to find the slope of a line, you then should be able to estimate the slope of the tangent line by using all the secant lines.

**Slope of a Secant Line for points $(a, f(a)$ and $(x, f(x))$.**




Slope: $m = \dfrac{f(x) - f(a)}{x - a}$

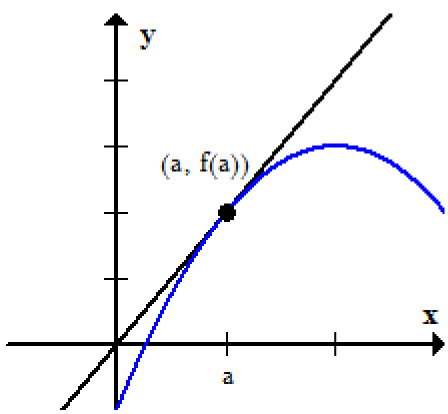
And you saw from above as the point $(x, f(x))$ moves closer to $(a, f(a))$, the secant lines get closer to looking like the tangent line. We spent the last 20 or so pages going over the concept of a limit. The slope of a tangent line is a limit of the slope of the secant lines.

Slope of Tangent Line

$$m_{\text{tangent}} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

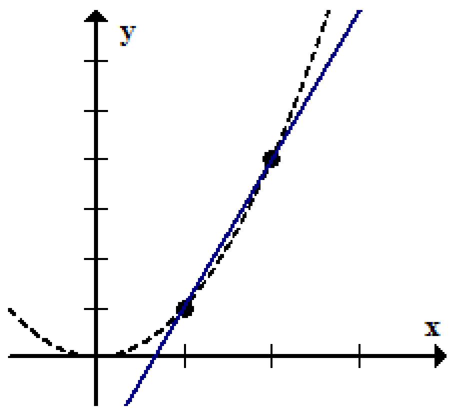
**Definition 2.1** (Tangent Line)**.** The tangent line to the curve $f(x)$ at point $(a, f(a))$ is the line through $(a, f(a))$ with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$



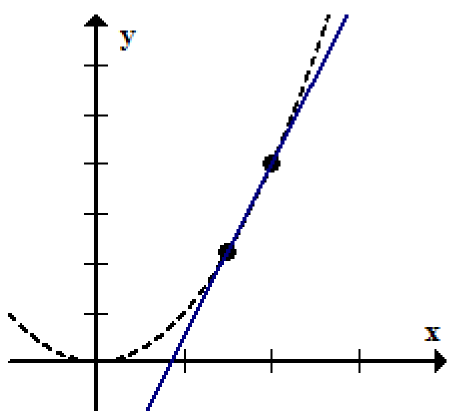

**Example 2.1.** Find an equation of the tangent line to $y = x^2$ at point (2,4)

1. Find the slope of the secant line joining $(1, 1)$ to $(2, 4)$.



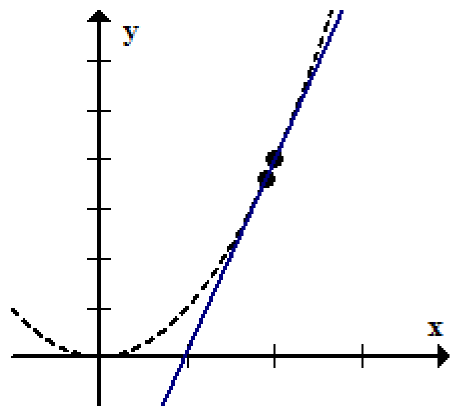

Slope: $m = \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{2 - 1} = 3$

2. Find the slope of the secant line joining $(1.5, 2.25)$ to $(2, 4)$.



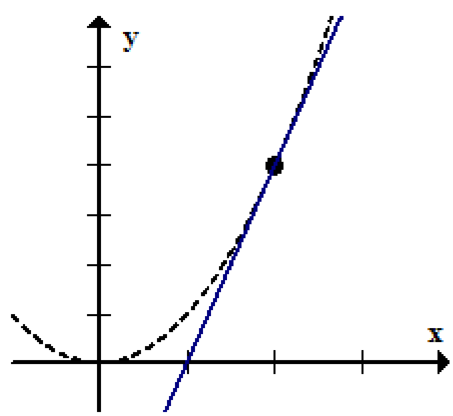

Slope: $m = \frac{f(2) - f(1.5)}{2 - 1.5} = \frac{4 - 2.25}{2 - 1.5} = 3.5$

3. Find the slope of the secant line joining $(1.9, 3.61)$ to $(2, 4)$.




Slope: $m = \dfrac{f(2) - f(1.9)}{2 - 1.9} = \dfrac{4 - 3.61}{2 - 1.9} = 3.9$

4. Find the slope of the secant line joining $(1.9, 3.61)$ to $(2, 4)$.




Slope: $m = \dfrac{f(2) - f(1.99)}{2 - 1.99} = \dfrac{4 - 3.9601}{2 - 1.99} = 3.99$

5. You see as our second point gets closer to $(2, 4)$, the slope of the secant lines appear to approach $m = 4$. Let's find the slope of the tangent line using our limit notation.

Slope of the Tangent Line:

$$\begin{aligned}
\lim_{x\to 2} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to 2} \frac{x^2-2^2}{x-2} \\
&= \lim_{x\to 2} \frac{x^2-4}{x-2} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\
&= \lim_{x\to 2} x+2 \\
&= 4
\end{aligned}$$

6. Find the equation of the tangent line at $(2,4)$.

   We need two things to find the equation of a line.

   (a) The Slope

   $$m = 4$$

   (b) A point

   $$(2,4)$$

   (c) Now we use the point-slope formula

   $$y - y_1 = m(x - x_1)$$

   $$y - 4 = 4(x-2)$$

   $$y = 4x - 4$$

**Example 2.2.** Find the slope of the tangent line on the curve $y = x^2$ at the point (0,0), (1,1), (-3, 9), (4,16).

Whoa! Do I have to use the limit every time to find the slope of the tangent line? The answer is no. You really only have to do it once.

Let's word this question a different way. Since I want the slope at a variety of points, let's find the slope of the tangent line at a generic point $(a, f(a))$.

$$\begin{aligned} m &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x\to a} \frac{x^2 - a^2}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x+1)}{x - a} \\ &= \lim_{x\to a} x + a \\ &= 2a \end{aligned}$$

If you want the slope of the tangent line at the point (0,0), let $a = 0$.

$$m = 2(0) = 0$$

If you want the slope of the tangent line at the point (-3,9), let $a = -3$.

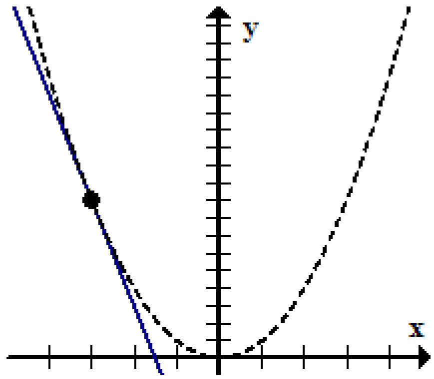
$$m = 2(-3) = -6$$

What's the equation of the tangent line through the point $(-3, 9)$.

$$y - y_1 = m(x - x_1)$$

$$y - 9 = -6(x - (-3))$$

$$y = -6x - 9$$




**Another Version of the Tangent Line Slope**

Instead of using points $(a, f(a))$ and $(x, f(x))$, we use $(a, f(a))$ and $(a+h, f(a+h))$.

- We let $x = a + h$.

- $x \to a$ is the same thing as $h \to 0$

- Instead of $f(x)$, we use $f(a+h)$.

- The slope formula is now either one

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\Rightarrow m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

**Example 2.3.** Using this new form, find the slope of the tangent line on the curve $y = x^2$ at the point (-3,9). BTW, we know it should be $m = -6$.

Let $a = -3$.

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} \\ &= \lim_{h\to 0} \frac{(-3+h)^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{(9 - 6h + h^2) - 9}{h} \\ &= \lim_{h\to 0} \frac{-6h + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(-6+h)}{h} \\ &= \lim_{h\to 0} -6 + h \\ &= -6 \end{aligned}$$

Wahoo! The slope matches. Math is awesome! ... too much?

**Example 2.4.** Find an equation of the tangent line on $y = \dfrac{3}{x}$ at the point (3,1). Use both limit versions of the slope formula.

1. Find the slope using $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$

$$\begin{aligned} m &= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3} \frac{\frac{3}{x}-1}{x-3} \\ ClearDenominators &= \lim_{x\to 3} \frac{\frac{3}{x}-1}{x-3}\cdot\frac{x}{x} \\ &= \lim_{x\to 3} \frac{3-x}{x(x-3)} \\ &= \lim_{x\to 3} \frac{-(x-3)}{x(x-3)} \\ &= \lim_{x\to 3} \frac{-1}{x} \\ &= -\frac{1}{3} \end{aligned}$$

2. Find the slope using $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{3+h}-1}{h} \\ ClearDenominators &= \lim_{h\to 0} \frac{\frac{3}{3+h}-1}{h}\cdot\frac{3+h}{3+h} \\ &= \lim_{h\to 0} \frac{3-(3+h)}{h(3+h)} \\ &= \lim_{h\to 0} \frac{-h}{h(3+h)} \\ &= \lim_{h\to 0} \frac{-1}{3+h} \\ &= -\frac{1}{3} \end{aligned}$$

3. We confirmed through both limit formulas that the slope is $m = -\frac{1}{3}$. To find the equation of the tangent line, we use the point-slope formula.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{3}(x-3)$$
$$y = -\frac{1}{3}x + 2$$

There is another interpretation of the slopes of the secant lines and the tangent line.

$$\frac{f(b)-f(a)}{b-a}$$

is the change in $f$ divided by the change in $x$. It represents the average rate of change of $f$ as $x$ goes from $a$ to $b$.

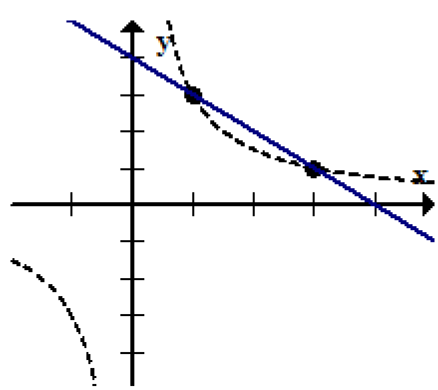
So what else can we call the slope of the tangent line at $x = a$? It represents the instantaneous rate of change at $x = a$.

**Example 2.5.** Let $f(x) = \frac{3}{x}$.

1. Find the average rate of change of $f(x) = \frac{3}{x}$ over the interval $[1, 4]$.

$$\frac{f(3)-f(1)}{3-1} = \frac{\frac{3}{3}-\frac{3}{1}}{2} = -1$$

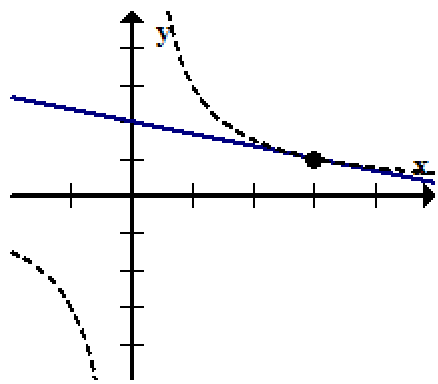
You an see the slope here



So what does an average rate of change of $-1$ mean? It means if $f$ continued to change at the same rate (in this case, -1), then for every unit $x$ increases, the function would decrease by 1 unit.

2. Find the instantaneous rate of change at $x = 3$.

We already did this. It's $m = -\frac{1}{3}$.




This means if $f$ continued at the same rate, then for every 3 units that $x$ increases, the function decreases by 1 unit.

But you see by looking at the graph, at different points of the graph, you'll have a different slope. So the slope is actually a function of $x$.

What does this mean? It means if you're given a function $f$, there is another function (denoted $f'$) then tells us the slope.

**Definition 2.2.** The derivative of a function $f$ at $x = a$, denoted $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f'$ is pronounced $f$ prime.

We refer to this formula as the **limit definition of a derivative**. Luckily, we will have shortcuts to finding these derivatives without limits.

We already used derivatives. Recall when $f(x) = x^2$. From a couple of examples ago, we found that the slope at the point $(a, f(a))$ is $2a$. Using our current terminology, if $f(x) = x^2$, the derivative at $x = a$ is $f'(a) = 2a$. This derivative function will tell us the slope at any point on $f(x)$.

**Example 2.6.** Find $f'(a)$ when $f(x) = 3 - 2x + 3x^2$.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{[3 - 2(a+h) + 3(a+h)^2] - [3 - 2a + 3a^2]}{h} \\
&= \lim_{h\to 0} \frac{3 - 2a - 2h + 3(a^2 + 2ah + h^2) - 3 + 2a - 3a^2}{h} \\
&= \lim_{h\to 0} \frac{3 - 2a - 2h + 3a^2 + 6ah + 3h^2 - 3 + 2a - 3a^2}{h} \\
&= \lim_{h\to 0} \frac{-2h + 6ah + 3h^2}{h} \\
&= \lim_{h\to 0} \frac{h(-2 + 6a + 3h)}{h} \\
&= \lim_{h\to 0} -2 + 6a + 3h \\
&= -2 + 6a
\end{aligned}$$

If $f(x) = 3-2x+3x^2$, we can find the slope at any point $(a, f(a))$ by using $f'(a) = -2+6a$.

1. Find the slope at (0,3).

$$f'(0) = -2 + 6(0) = -2$$

2. Find the slope at (-2, 19).

$$f'(-2) = -2 + 6(-2) = -14$$

**Example 2.7.** Find the tangent line on $y = \dfrac{x+4}{x-2}$ at (4,4).

First, let's find the slope. We'll use the following limit formula,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Before starting, note that we already know $a$. If you're trying to find the slope at only one point, you might as well use it.

$$\begin{aligned}
m &= \lim_{h\to 0} \frac{\frac{(4+h)+4}{(4+h)-2} - \frac{4+4}{4-2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{h+8}{h+2} - 4}{h} \\
\text{Clear the Denominators} &\rightarrow \lim_{h\to 0} \frac{\frac{h+8}{h+2} - 4}{h} \cdot \frac{h+2}{h+2} \\
&= \lim_{h\to 0} \frac{h+8-4(h+2)}{h(h+2)} \\
&= \lim_{h\to 0} \frac{h+8-4h-8}{h(h+2)} \\
&= \lim_{h\to 0} \frac{-3h}{h(h+2)} \\
&= \lim_{h\to 0} \frac{-3}{h+2} \\
&= -\frac{3}{2}
\end{aligned}$$

Now, use the point-slope formula to find the equation of the tangent line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{2}(x - 4) \\ y &= -\frac{3}{2}x + 10 \end{aligned}$$

**Velocity**

If $s$ is a position function, then the average velocity over $t = a$ to $t = a + h$ is

$$v_{\text{avg}} = \frac{s(a+h) - s(a)}{h}$$

As $h \to 0$; i.e., $\lim\limits_{h \to 0} \dfrac{s(a+h) - s(a)}{h}$, this is called **instantaneous velocity** at $t = a$.

Think of $s(t)$ as the position of an object moving in one dimension (moving left or right on the $x$-axis or straight or down).

**Example 2.8.** Suppose a ball is dropped from an observation deck, 450 $m$ above ground.

1. What is the velocity after 5 seconds? Use $s(t) = 4.9t^2$.
2. Find $v(a)$ - the velocity function

$$\begin{aligned}
v(a) &= \lim_{h\to 0} \frac{s(a+h)-s(a)}{h} \\
&= \lim_{h\to 0} \frac{4.9(a+h)^2-4.9a^2}{h} \\
&= \lim_{h\to 0} \frac{4.9(a^2+2ah+h^2)-4.9a^2}{h} \\
&= \lim_{h\to 0} \frac{9.8ah+4.9h^2}{h} \\
&= \lim_{h\to 0} \frac{h(9.8a+4.9h)}{h} \\
&= \lim_{h\to 0} 9.8a+4.9h \\
&= 9.8a
\end{aligned}$$

So the velocity after 5 seconds is

$$v(5) = 9.8(5) = 49\ m/s$$

3. Find the velocity when it hits the ground.

   (a) How long will it take to hit the ground?

   Since we are 450 m above ground, solves $s(t) = 4.9t^2 = 450$.

$$4.9t^2 = 450$$

$$4.9t^2 - 450 = 0$$

$$t^2 = 450/4.9$$

$$t = 9.58 \text{ seconds}$$

(b) The velocity at 9.58 seconds is

$$v(9.58) = 9.8(9.58) = 93.88\ m/s$$