Business Calculus

Brian E. Veitch

Northern Illinois University

February 8, 2014

Rational Functions

Definition 1.1

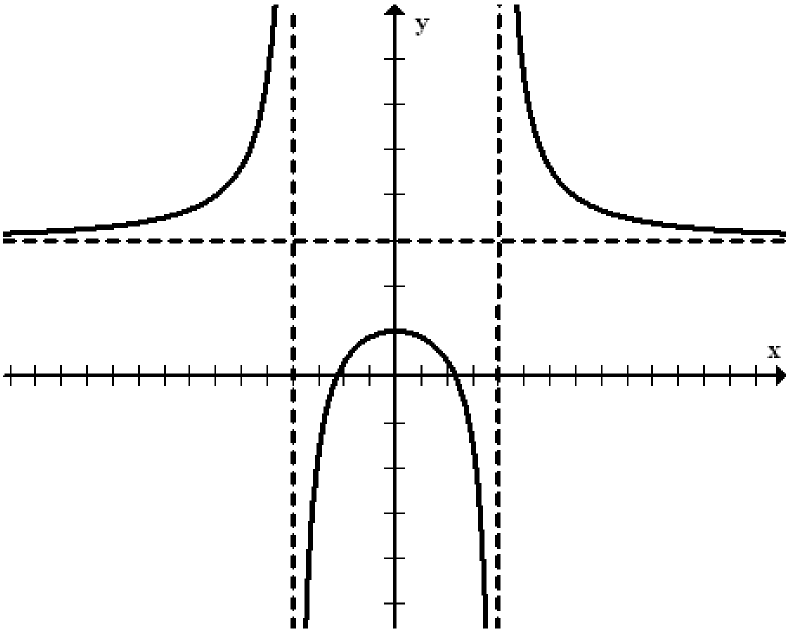
A **rational function** is a function f that can be written as

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. The domain of f is all x where $Q(x) \neq 0$.

Examples:

Vertical and Horizontal Asymptotes



Definition 1.2

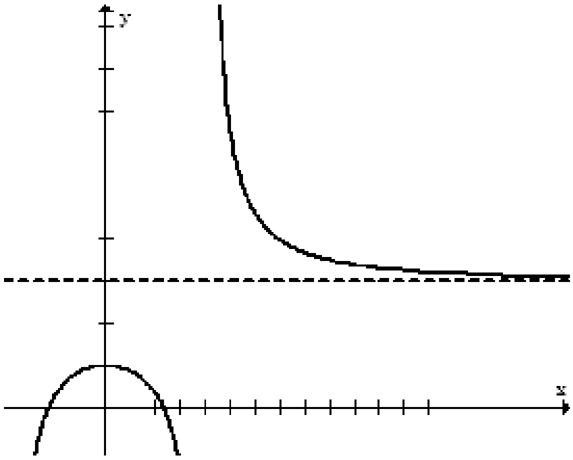
The line $x = a$ is a **vertical asymptote** if any of the following are true:

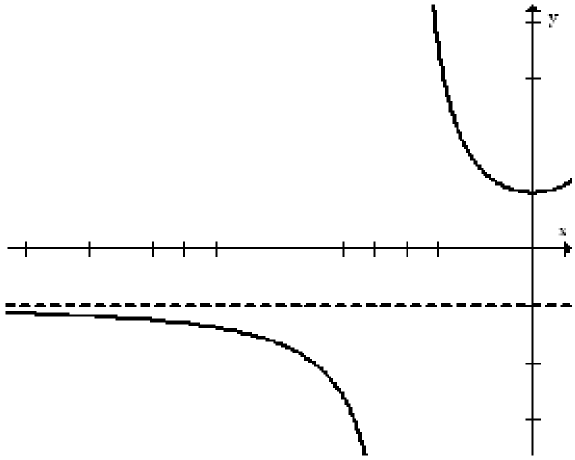
$$\lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty$$

From algebra, you find **vertical asymptotes** by finding all x values that make the denominator 0 (**after the rational function is reduced**).

Find the vertical asmptotes for the following two functions:

$$f(x) = \frac{x}{x^2 - 4} \qquad g(x) = \frac{x-2}{x^2-4}$$



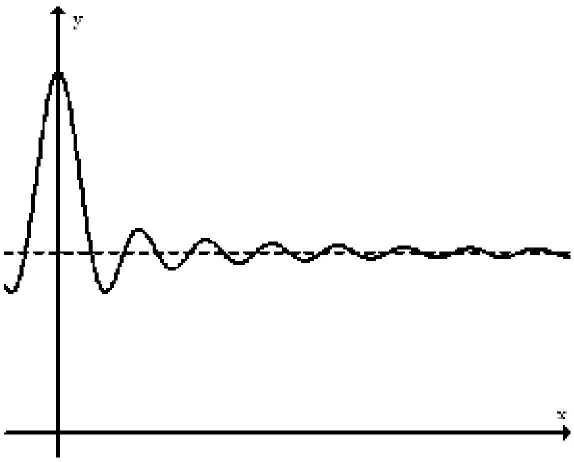


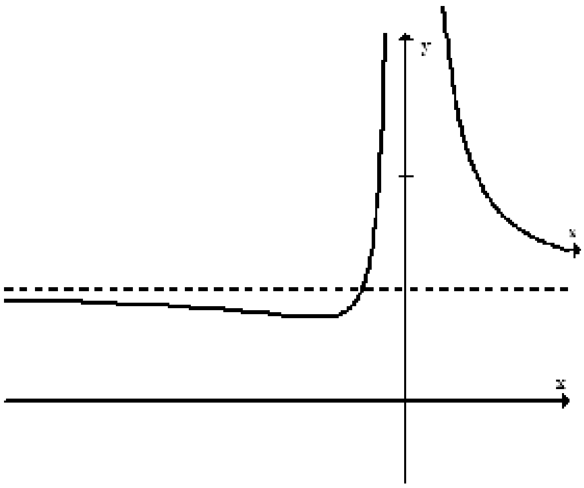
Definition 1.3

The line $y = b$ is a **horizontal asymptote** if either or both of the following is true:

$$\lim_{x\to-\infty} f(x) = b \text{ or } \lim_{x\to\infty} f(x) = b$$

The graph of a rational function may or may not cross a horizontal asymptote. They occur when the degree of the demoninator is bigger or equal to the degree of the numerator.





Theorem 1.4

If $f(x) = \dfrac{a}{b^n}$, *then* $\lim_{x\to\pm\infty} \dfrac{a}{b^n} = 0$.

Example 1.5

Determine the **Horizontal Asymptote** of by evaluating the following limits:

1. $\displaystyle\lim_{x\to\infty} \frac{5x-1}{2x+7}$

2. $\displaystyle\lim_{x\to-\infty} \frac{3x^4+x^2-8}{2x^4-x^3}$

3. $\displaystyle\lim_{x\to\infty} \frac{x^2-1}{1-7x^4}$

A useful tool for finding Horizontal Asymptotes

Theorem 1.6

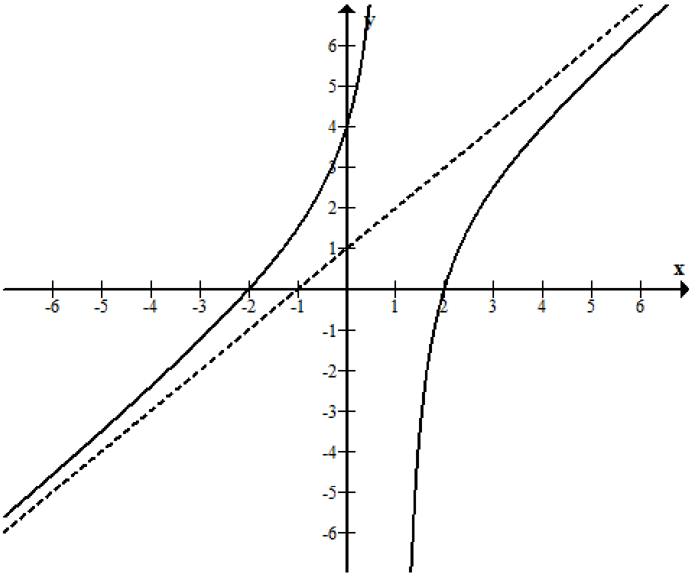
If $f(x) = \dfrac{ax^n+\dots}{bx^n+\dots}$, *then*

$$\lim_{x\to\pm\infty} \frac{ax^n+\dots}{bx^n+\dots} = \frac{a}{b}$$

If $f(x) = \dfrac{ax^m+\dots}{bx^n+\dots}$, *where* $m < n$, *then*

$$\lim_{x\to\pm\infty} \frac{ax^m+\dots}{bx^n+\dots} = 0$$

Slant Asymptotes



A linear asymptote that is neither vertical nor horizontal is called a slant asymptote or an oblique asymptote. For any rational function of the form $f(x) = \dfrac{p(x)}{q(x)}$, a slant asymptote occurs when the degree of $p(x)$ [the top] is exactly 1 more than the degree of $q(x)$ [the bottom]. A graph can cross a slant asymptote. The line $y = mx + b$ is the oblique asymptote of $f(x)$.

Example 1.7

Let $f(x) = \dfrac{x^2 - 4}{x - 1}$. Find all asymptotes.

Summary of Curve Sketching

1. Domain of $f(x)$
2. x and y intercepts
   1. x-intercepts occur when $f(x) = 0$
   2. y-intercept occurs when $x = 0$
3. Find the asymptotes (vertical, horizontal / slant). Graph them.
4. Find $f'(x)$
   1. Find the critical values, all x-values where $f'(x) = 0$ or when $f'(x)$ does not exist. **These are potential local extrema.**
   2. Find increasing / decreasing intervals using numberline
   3. Find local maximums / minimums (if any exist). Remember to write them as points.
      1. Local Max at $x = c$: $f'(x)$ changes from (+) to (-) at $x = c$.
      2. Local Min at $x = c$: $f'(x)$ changes from (-) to (+) at $x = c$.

5. Plot them
6. Find $f''(x)$
   1. Find all x-values where $f''(x) = 0$ or when $f''(x)$ does not exist. **These are potential inflection points.**
   2. Find intervals of concavity using the number line
   3. Find points of inflection
      1. Must be a place where concavity changes
      2. The point must exist (i.e, can't be an asymptote, discontinuity)
   4. Plot them
7. Sketch

Example 1.8

Sketch the graph $f(x) = -\dfrac{3}{x-4}$.

Example 1.9

Sketch the graph $f(x) = \dfrac{2x^2}{x^2 - 1}$.

Example 1.10

Sketch the graph $f(x) = \dfrac{8}{x^2 - 4}$.

Example 1.11

Sketch the graph $f(x) = -\dfrac{x^2 - 2x + 5}{x - 1}$.

Chapter 2 - Applications of Differentiation 2.3 - Asymptotes and Rational Functions

Example 1.12

Sketch the graph $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$, $f'(x) = -\frac{x+2}{x^3}$, and $f''(x) = \frac{2(x+3)}{x^4}$.